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Joining models with commutative orthogonal block structure

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A B S T R A C T

Mixed linear models are a versatile and powerful tool for analysing data collected in experiments in several areas. A mixed model is a model with orthogonal block structure, OBS, when its variance–covariance matrix is of all the positive semi-definite linear combinations of known pairwise orthogonal orthogonal projection matrices that add up to the identity matrix. Models with commutative orthogonal block structure, COBS, are a special case of OBS in which the orthogonal projection matrix on the space spanned by the mean vector commutes with the variance–covariance matrix.
Using the algebraic structure of COBS, based on Commutative Jordan algebras of symmetric matrices, and the Cartesian product we build up complex models from simpler ones through joining, in order to analyse together models obtained independently. This commutativity condition of COBS is a necessary and sufficient condition for the least square estimators, LSE, to be best linear unbiased estimators, BLUE, whatever the variance components. Since joining COBS we

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obtain new COBS, the good properties of estimators hold for the joined models.

1. Introduction

Mixed models are a versatile and powerful tool for analysing data collected in experiments, see for example [12], and, over the years, they have been applied to several areas such as biological and medical research, animal and human genetics, agriculture or industry.

A mixed model whose variance–covariance matrix is of all the positive semi-definite linear combinations,

$$\sum_{j=1}^{m} \gamma_j \boldsymbol{Q}_j$$

of known pairwise orthogonal orthogonal projection matrices, POOPM, $\boldsymbol{Q}_1 \dots \boldsymbol{Q}_m$, that up to $\boldsymbol{I}_n$, is a model with orthogonal block structure, OBS. These models, introduced by Nelder [16,17], have been intensively studied [10,13] and continue to play a central part in the theory of randomised block designs [2,3].

OBS in which the matrices $\boldsymbol{Q}_1 \dots \boldsymbol{Q}_m$ commute with $\boldsymbol{T}$, the orthogonal projection matrix on the space spanned by the mean vector, are called models with commutative orthogonal block structure, COBS. This special class of OBS, was introduced in [9] and has also been considered by Santos et al. [19], Nunes et al. [18], Carvalho et al. [4], Ferreira et al. [7], Carvalho et al. [5] and Bailey et al. [1]. Therefore COBS are models with OBS whose variance–covariance matrix commutes with the orthogonal projection matrix on the space spanned by the mean vector. This commutativity condition is a necessary and sufficient condition for the least square estimators, LSE, to be best linear unbiased estimators, BLUE, whatever the variance components [23].

In order to build up complex models from simple ones, Mexia et al. [15] introduced models crossing and models nesting, two operations between models based on the binary operations on CJAS, Kronecker product of CJAS and the restricted Kronecker product of CJAS, introduced in Fonseca et al. [8].

Now we introduce models joining, a possible alternative to models crossing and models nesting, with the same purpose to analyse together models obtained independently.

Let $\boldsymbol{y}(1) \dots \boldsymbol{y}(n)$ be the observations vectors of n models with null cross-covariance matrices, then

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y}(1) \\ \vdots \\ \boldsymbol{y}(n) \end{bmatrix}$$

will be the observations vector of the model obtained joining the initial models. We will consider this operation for models with commutative orthogonal block structure, COBS.

Since the technique used for building joined models rests on the algebraic structures of COBS and the Cartesian product of CJAS, in the next section we present some results on commutative Jordan algebras that will be useful on studying both the algebraic structures of OBS and COBS and in carrying out models joining.

2. Commutative Jordan algebras of symmetric matrices

Jordan algebras, JA, were introduced by Jordan et al. [11] in their paper devoted to the axiomatic foundation of quantum mechanics, and rediscovered later by Seely [21] who used them to solve problems in Linear Statistical Inference, calling them quadratic vector spaces.

A CJAS is a linear space constituted by symmetric matrices that commute containing the squares of its matrices. Seely [22] showed that every CJAS, A, has an unique basis, the principal basis, $pb(A)$, constituted by POOPM.

Let A be a CJAS, of $n \times n$ matrices, with principal basis $pb(A) = Q = \{\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m\}$. Given $\boldsymbol{M} \in A$, we have

$$\boldsymbol{M} = \sum_{j=1}^{m} b_j \boldsymbol{Q_j} = \sum_{j \in \mathfrak{C}(\boldsymbol{M})} b_j \boldsymbol{Q_j}$$

with $\mathfrak{C}(\boldsymbol{M}) = \{j : b_j \neq 0\}$, $j = 1, \ldots, m$.

The orthogonal projection matrices, OPM, belonging to a CJAS, A, are sums of matrices of $pb(A)$. Since $\{\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m\} = pb(A)$ has m matrices, the CJAS A, as a linear subspace, has dimension $dim(A) \subseteq m$. Thus, considering the $\boldsymbol{0}_{n\times n}$ matrices as an OPM on $\{\boldsymbol{0}^n\}$, there can be 2^n OPM in A, as much as the distinct sums of matrices of $pb(A)$, once each of the sums corresponds to a sub-set of $\overline{\overline{m}} = \{1, \ldots, m\}$. Given $C \subseteq \overline{\overline{m}}$ and $\boldsymbol{Q}(C) = \sum_{j\in C} \boldsymbol{Q}_j$, we will have $rank(\boldsymbol{Q}(C)) = \sum_{j\in C} g_j$, where $g_j = rank\, \boldsymbol{Q}_j$, $j = 1, \ldots, m$.

We also see that if, with $\boldsymbol{Q} \in A$, we have $rank(\boldsymbol{Q}) = 1$ then we must have $\boldsymbol{Q} \in pb(A)$. Namely, with $\boldsymbol{J}_n = \boldsymbol{1}_n\boldsymbol{1}_n^T$, if $\boldsymbol{Q} = \frac{1}{n}\boldsymbol{J}_n \in A$ we put $\boldsymbol{Q}_1 = \boldsymbol{Q}$ and say that A is a regular CJAS.

If the CJAS A is complete, this is, when A contains invertible matrices, we must have $\sum_{j=1}^{m} \boldsymbol{Q}_j = \boldsymbol{I}_n$, so $\sum_{j=1}^{m} g_j = n$, thus the matrices in the principal basis of a complete CJAS add up to $\boldsymbol{I}_n$.

Given $\boldsymbol{M} = \sum_{j=1}^{m} b_j \boldsymbol{Q}_j$, with $b_j \neq 0$, the b_j, $j = 1, \ldots, m$, will be the eigenvalues of $\boldsymbol{M}$ with multiplicities g_j, $j = 1, \ldots, m$, so the determinant of matrix $\boldsymbol{M}$ will be

$$det(\boldsymbol{M}) = \prod_{j=1}^{m} b_j^{g_j}$$

and

$$\boldsymbol{M}^{-1} = \sum_{j=1}^{m} b_j^{-1} \boldsymbol{Q}_j$$

whenever $\boldsymbol{M}$ is invertible.

Given the family $M = \{\boldsymbol{M}_1, \dots, \boldsymbol{M}_w\}$, of matrices of A, we will have

$$\boldsymbol{M}_i = \sum_{j=1}^{m} b_{i,j} \boldsymbol{Q}_j, \quad i = 1, \dots, w$$

and $\boldsymbol{B} = [b_{i,j}]$ will be the transition matrix between M and Q, $M \backslash Q$. The matrices in M are linearly independent when and only when the row vectors of $\boldsymbol{B}$ are linearly independent, and $rank(A) = m$. If $w = m$ and the matrices $\boldsymbol{M}_1, \dots, \boldsymbol{M}_m$ are linearly independent, the m row vectors of $\boldsymbol{B}$ will be linearly independent, thus $\boldsymbol{B}$ will be $m \times m$ and $rank(\boldsymbol{B}) = m$.

Having $\boldsymbol{M}_i = \sum_{j=1}^{m} b_{i,j} \boldsymbol{Q}_j$, $i = 1, \dots, m$, with $\boldsymbol{M}_i = [m_{i,l,h}]$ and $\boldsymbol{Q}_j = [q_{j,l,h}]$ we have, for every pair (l, h), $m_{i,l,h} = \sum_{j=1}^{m} b_{i,j} q_{j,l,h}$, $i = 1, \dots, m$, $l = 1, \dots, n$, $h = 1, \dots, n$, this is, with $\dot{\boldsymbol{m}}(l, h)[\dot{\boldsymbol{q}}(l, h)]$ the vector with components $m_{1,l,h}, \dots, m_{m,l,h}[q_{1,l,h}, \dots, q_{m,l,h}]$, $\dot{\boldsymbol{m}}(l, h) = \boldsymbol{B} \dot{\boldsymbol{q}}(l, h)$, $l = 1, \dots, n$, $h = 1, \dots, n$.

Since $\boldsymbol{B}$ is invertible so is $\boldsymbol{B}^T$ and we have

$$\dot{\boldsymbol{q}}(l, h) = \left(\boldsymbol{B}^T\right)^{-1} \dot{\boldsymbol{m}}(l, h), \quad l = 1, \dots, n, \; h = 1, \dots, n,$$

so, with $\boldsymbol{B}^{-1} = [b^{j,i}]$, we have

$$q_{j,l,h} = \sum_{i=1}^{m} b^{j,i} m_{i,l,h}, \quad l = 1, \dots, n, \; h = 1, \dots, n, \; j = 1, \dots, m$$

which is the same as

$$\boldsymbol{Q}_j = \sum_{i=1}^{m} b^{j,i} \boldsymbol{M}_i,$$

$j = 1, \dots, m$ and M will be a basis for A.

Now, the matrices of $M = \{\boldsymbol{M}_1, \dots, \boldsymbol{M}_m\}$ commute, see Schott (1997) [20], if and only if they are diagonalised by the same orthogonal matrix, $\boldsymbol{P}^0$. We then have $M \subset V(\boldsymbol{P}^0)$, with $V(\boldsymbol{P}^0)$ the family of matrices diagonalised by $\boldsymbol{P}^0$. Since $V(\boldsymbol{P}^0)$ is a CJAS we see that a family of $n \times n$ symmetric matrices is contained in a CJAS if and only if they commute. Since the intersection of CJAS gives CJAS there will be a minimum CJAS containing M, whose matrices commute, this will be the CJAS $A(M)$ generated by M.

Namely if D is a family of POOPM, $A(D)$ will have D as principal basis since the CJAS D must contain the CJAS constituted by the linear combinations of the matrices.

If the $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_w$ commute and are diagonalised by the orthogonal matrix $\boldsymbol{P}^0$ the row vectors $\underline{\alpha}_1, \ldots \underline{\alpha}_n$ of $\boldsymbol{P}^0$ will be eigenvectors for the matrices of M.

3. Models with commutative orthogonal block structure

The mixed model

$$\boldsymbol{y} = \sum_{i=0}^{w} \boldsymbol{X}_i \boldsymbol{\beta}_i$$

where $\boldsymbol{\beta}_0$ is fixed and $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_w$ are independent random vectors with null mean vectors and variance–covariance matrices $\sigma_1^2 \boldsymbol{I}_{c_1} \ldots \sigma_w^2 \boldsymbol{I}_{c_w}$, where $c_i = rank(\boldsymbol{X_i})$, $i = 1, \ldots, k$, will have mean vector $\boldsymbol{\mu} = \boldsymbol{X}_0 \boldsymbol{\beta}_0$ and variance–covariance matrix $\boldsymbol{V} = \sum_{i=1}^{w} \sigma_i^2 \boldsymbol{M}_i$, where $\boldsymbol{M}_i = \boldsymbol{X}_i \boldsymbol{X}_i^T$, $i = 1, \ldots, w$.

When the matrices of $M^0 = \{\boldsymbol{M}_1 \ldots \boldsymbol{M}_w\}$ commute, they will generate a CJAS, A^0, with principal basis $Q^0 = \{\boldsymbol{Q}_1^0, \ldots, \boldsymbol{Q}_{m^0}^0\}$.

If the $\mathbb{R}^n$ is the range space of $[\boldsymbol{X_1}, \ldots, \boldsymbol{X_w}]$, the $n \times n$ matrix $\boldsymbol{M_i} = [\boldsymbol{X_1}, \ldots, \boldsymbol{X_w}][\boldsymbol{X_1}, \ldots, \boldsymbol{X_w}]^T$ will belong to A^0 and, having rank n, will be invertible so A^0 will be complete.

The space, Ω, spanned by $\boldsymbol{\mu}$ will be $R(\boldsymbol{X}_0)$ so, the orthogonal projection matrix on Ω will be

$$\boldsymbol{T} = \boldsymbol{X}_0 \left(\boldsymbol{X}_0^T \boldsymbol{X}_0\right)^{\dagger} \boldsymbol{X}_0^T = \boldsymbol{X}_0 \boldsymbol{X}_0^{\dagger},$$

where $\dagger$ indicates Moore–Penrose inverse.

Now we will have

$$\boldsymbol{M_i} = \sum_{j=1}^{m^0} b_{i,j}^0 \boldsymbol{Q}_j^0, \quad i = 1, \ldots, w,$$

and so

$$\begin{aligned} \boldsymbol{V} &= \sum_{i=1}^{w} \sigma_i^2 \boldsymbol{M}_i = \sum_{i=1}^{w} \sigma_i^2 \sum_{j=1}^{m^0} b_{i,j}^0 \boldsymbol{Q}_j^0 \\ &= \sum_{j=1}^{m^0} \left(\sum_{i=1}^{w} b_{i,j}^0 \sigma_i^2 \right) \boldsymbol{Q}_j^0 = \sum_{j=1}^{m^0} \gamma_j^0 \boldsymbol{Q}_j^0 \end{aligned}$$

with

$$\gamma_j^0 = \sum_{i=1}^{w} b_{i,j}^0 \sigma_i^2, \quad j = 1, \dots, m^0.$$

When $M^0 = \{\boldsymbol{M}_1 \dots \boldsymbol{M}_w\}$ is a basis for A^0 we have $w = m^0$ and the transition matrix $\boldsymbol{B}^0 = [b_{i,j}^0]$ is an $m^0 \times m^0$ invertible matrix. Then the variance–covariance matrices of the mixed model will be all positive semi-definite linear combinations of the matrices $\boldsymbol{Q}_1^0, \dots, \boldsymbol{Q}_{m^0}^0$ and the model will have OBS.

Let us now assume that $\boldsymbol{T}$ commutes with the matrices $\boldsymbol{M}_1, \dots, \boldsymbol{M}_w$, then the matrices of $M^* = \{\boldsymbol{T}, \boldsymbol{M}_1 \dots \boldsymbol{M}_w\}$ will generate a CJAS that contains A^0, thus containing $\boldsymbol{T}$ and the $\boldsymbol{Q}_1^0, \dots, \boldsymbol{Q}_{m^0}^0$, so $\boldsymbol{T}\boldsymbol{Q}_j^0 = \boldsymbol{Q}_j^0\boldsymbol{T}$, $j = 1, \dots, m^0$, and the model will have COBS.

We point out that the principal basis of a CJAS A is constituted by the nonnull matrices $\boldsymbol{Q}_j^0\boldsymbol{T}$ and $\boldsymbol{Q}_j^0\boldsymbol{T}^{\boldsymbol{C}}$, with $\boldsymbol{T}^{\boldsymbol{C}} = \boldsymbol{I}_n - \boldsymbol{T}$ [15].

Let $z^0 \geq 0$ be the number of matrices of Q^0 such that $\boldsymbol{Q}_j^{\boldsymbol{0}}\boldsymbol{T} = \boldsymbol{Q}_j^{\boldsymbol{0}}$ and $z - z^0 \geq 0$ be the number of matrices of Q^0 such that $\boldsymbol{Q}_j^{\boldsymbol{0}}\boldsymbol{T} \neq \boldsymbol{0}_{n\times n}$ and $\boldsymbol{Q}_j^{\boldsymbol{0}}\boldsymbol{T} \neq \boldsymbol{Q}_j^{\boldsymbol{0}}$. We can always order the matrices in Q^0 and Q to have

$$\begin{cases} \boldsymbol{Q}_j = \boldsymbol{Q}_j^0, & j = 1, \dots, z^0 \ \left(\text{if } z^0 > 0\right) \\ \boldsymbol{Q}_j = \boldsymbol{Q}_j^0\boldsymbol{T}, & j = z^0 + 1, \dots, z \\ \boldsymbol{Q}_j = \boldsymbol{Q}_{j-z+z^0}^0\boldsymbol{T}^C, & j = z + 1, \dots, m \end{cases}$$

Then we will have $\boldsymbol{T} = \sum_{j=1}^{z} \boldsymbol{Q}_{\boldsymbol{j}}$ and $m^0 = m - z + z^0$ as well as

$$\begin{cases} \boldsymbol{Q}_j^0 = \boldsymbol{Q}_j, & j = 1, \dots, z^0 \ \left(\text{if } z^0 > 0\right) \\ \boldsymbol{Q}_j^0 = \boldsymbol{Q}_j + \boldsymbol{Q}_{j+z-z^0}, & j = z^0 + 1, \dots, z \\ \boldsymbol{Q}_j^0 = \boldsymbol{Q}_{j+z-z^0}, & j = z + 1, \dots, m^0 \end{cases}$$

Since

$$\boldsymbol{V} = \sum_{j=1}^{m^0} \gamma_j^0 \boldsymbol{Q}_j^0 = \sum_{l=1}^{m} \gamma_l \boldsymbol{Q}_l,$$

we get

$$\begin{cases} \gamma_j^0 = \gamma_j, & j = 1, \dots, z^0 \ \left(\text{if } z^0 > 0\right) \\ \gamma_j^0 = \gamma_j = \gamma_{j+z-z^0}, & j = z^0 + 1, \dots, z \\ \gamma_j^0 = \gamma_{j+z-z^0}, & j = z + 1, \dots, m^0 \end{cases}$$

Likewise, from $\boldsymbol{M}_i = \sum_{j=1}^{m^0} b_{i,j}^0 \boldsymbol{Q}_j^0 = \sum_{j=1}^{m} b_{i,j} \boldsymbol{Q}_j$ we get

$$\begin{cases} b_{i,j}^0 = b_{i,j}, & j = 1, \dots, z^0, i = 1 \dots, w, \ \left(\text{if } z^0 > 0\right) \\ b_{i,j}^0 = b_{i,j} = b_{i,j+z-z^0}, & j = z^0 + 1, \dots, z, i = 1 \dots, w \\ b_{i,j}^0 = b_{i,j+z-z^0}, & j = z + 1, \dots, m^0, i = 1 \dots, w. \end{cases}$$

Thus $\boldsymbol{B^0} = [b^0_{i,j}]$ is sub matrix of $\boldsymbol{B} = [b_{i,j}]$, since every column of $\boldsymbol{B}^0$ is column of $\boldsymbol{B}$. Moreover the column of $\boldsymbol{B}$ with indexes j and $j+z$, $j = z^0+1, \dots, z$, will be identical, and every column of $\boldsymbol{B}$ is equal to a column of $\boldsymbol{B}^0$ so $R(\boldsymbol{B}^0) = R(\boldsymbol{B})$ and $rank(\boldsymbol{B}^0) = rank(\boldsymbol{B})$.

With $\boldsymbol{T} = \sum_{j=1}^{z} \boldsymbol{Q}_j$, let us put

$$\boldsymbol{\gamma}_1 = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_z \end{bmatrix},$$

$$\boldsymbol{\gamma}_2 = \begin{bmatrix} \gamma_{z+1} \\ \vdots \\ \gamma_m \end{bmatrix},$$

$$\boldsymbol{\sigma^2} = \begin{bmatrix} \sigma_1^2 \\ \vdots \\ \sigma_m^2 \end{bmatrix}$$

then, with the partition $\boldsymbol{B} = [\boldsymbol{B}_1 \ \boldsymbol{B}_2]$ where $\boldsymbol{B}_1$ has z columns, we have the vectors of canonical variance components

$$\boldsymbol{\gamma}_l = \boldsymbol{B}_l^T \boldsymbol{\sigma}^2, \quad l = 1, 2.$$

When $\boldsymbol{B_2}$ is horizontally free the column vectors of $\boldsymbol{B}_2^T$ are linearly independent and we obtain, see e.g. [14], $\boldsymbol{\sigma}^2 = (\boldsymbol{B}_2^T)^{\dagger} \boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_1 = \boldsymbol{B}_1^T (\boldsymbol{B}_2^T)^{\dagger} \boldsymbol{\gamma}_2$ so that we may estimate $\boldsymbol{\sigma}^2$ and $\boldsymbol{\gamma}_1$ through $\boldsymbol{\gamma}_2$. Then the relevant parameters for the random effects part of the model, $\boldsymbol{\gamma}_2$ and $\boldsymbol{\sigma}^2$, determine each other. Thus the random effects part of the model segregates as a sub-model and we say that there is segregation.

If $z^0 = 0$ the columns of $\boldsymbol{B}_1$ are identical to the first z columns of $\boldsymbol{B}_2$, and the corresponding components of $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ are identical. We then say that there is matching.

We can also define z^0 as the number of matrices in $\boldsymbol{Q}$ that belong to $\boldsymbol{Q}^0$, so this number will not depend on the ordering of the matrices on the two basis. As we shall see this possibility will be useful later on.

Let the row vectors of $\boldsymbol{A}_j$ constitute an orthogonal basis for $R(\boldsymbol{Q}_j)$ and put $\boldsymbol{X}_{0,j} = \boldsymbol{A}_j \boldsymbol{X}_0$, $= 1, \dots, m$, as well as $\boldsymbol{\mu}_j = \boldsymbol{A}_j \boldsymbol{\mu}$, $j = 1, \dots, m$. Now we have $\boldsymbol{Q}_j = \boldsymbol{A}_j^T \boldsymbol{A}_j$, $j = 1, \dots, m$ and

$$\boldsymbol{\mu} = \sum_{j=1}^{m} \boldsymbol{Q}_j \boldsymbol{\mu} = \sum_{j=1}^{m} \boldsymbol{A}_j^T \boldsymbol{A}_j \boldsymbol{\mu} = \sum_{j=1}^{m} \boldsymbol{A}_j^T \boldsymbol{\mu}_j$$

and so, whatever the $l \times n$ matrix $\boldsymbol{U}$, the estimable vector

$$\boldsymbol{\Psi} = \boldsymbol{U} \boldsymbol{\mu}$$

may be written as

$$\boldsymbol{\Psi} = \sum_{j=1}^{m} \boldsymbol{U}_j \boldsymbol{\mu}_j,$$

with $\boldsymbol{U}_j = \boldsymbol{U}\boldsymbol{A}_j^T$, $j = 1, \dots, m$.

The subvectors $\boldsymbol{Y}_j$ have mean vectors $\boldsymbol{\mu}_j$ and variance–covariance matrices

$$\boldsymbol{V}_j = \boldsymbol{A}_j \left(\sum_{j=1}^{m} \gamma_j \boldsymbol{Q}_j \right) \boldsymbol{A}_j^T = \gamma_j \boldsymbol{I}_{g_j},$$

with $g_j = rank(\boldsymbol{Q}_j)$, $j = 1, \dots, m$.

4. Models joining

Joining the mixed models

$$\boldsymbol{y}(l) = \sum_{i=0}^{w(l)} \boldsymbol{X}_i(l)\boldsymbol{\beta}_i(l), \quad l = 1, \dots, h,$$

where $\boldsymbol{\beta}_0(l)$, $l = 1, \dots, h$, were fixed and the $\boldsymbol{\beta}_1(l), \dots, \boldsymbol{\beta}_w(l)$, $l = 1, \dots, h$, had null mean vectors, null cross covariance matrices and variance–covariance matrices $\sigma^2(l)\boldsymbol{I}_{c_i(l)}$, where $c_i(l) = rank(\boldsymbol{X_i}(l))$, $i = 1, \dots, w(l)$, $l = 1, \dots, h$, we get the model

$$\boldsymbol{y} = \sum_{i=0}^{w} \boldsymbol{X}_i \boldsymbol{\beta}_i,$$

where, with $D(\boldsymbol{X}_0(1) \boldsymbol{\ldots} \boldsymbol{X}_0(h))$ indicating a blockwise diagonal matrix with principal blocks $\boldsymbol{X}_0(1) \boldsymbol{\ldots} \boldsymbol{X}_0(h)$,

$$\begin{cases} \boldsymbol{X}_0 = D(\boldsymbol{X}_0(1) \boldsymbol{\ldots} \boldsymbol{X}_0(h)) \\ \boldsymbol{\beta}_0 = [\boldsymbol{\beta}_0(1)^{\mathrm{T}} \boldsymbol{\ldots} \boldsymbol{\beta}_0(h)^{\mathrm{T}}]^{\mathrm{T}}. \end{cases}$$

Moreover putting $\overline{w}(0) = 0$ and $\overline{w}(l) = \sum_{k=1}^{l} w(k)$, $l = 1, \dots, h$, we also take, when $\overline{w}(l-1) < i \leq \overline{w}(l)$, $l = 1, \dots, h$,

$$\begin{cases} \boldsymbol{X}_i = D(\mathbf{0}_{n_1 \times n_l} \cdots \boldsymbol{X}_{i-\overline{w}(l-1)}(l) \cdots \mathbf{0}_{n_h \times n_l}) \\ \boldsymbol{\beta}_i = D(\mathbf{0}_{n_1 \times n_l} \cdots \boldsymbol{\beta}_{i-\overline{w}(l-1)}(l) \cdots \mathbf{0}_{n_h \times n_l}) \end{cases}$$

thus

$$\boldsymbol{M}_i = \boldsymbol{X}_i \boldsymbol{X}_i^T = D\big(\mathbf{0}_{n(1)\times n(1)} \dots \boldsymbol{M}_{i-\overline{w}(l-1)}(l) \dots \mathbf{0}_{n(h)\times n(h)}\big).$$

Let us now assume that the initial models are COBS. The principal basis for the CJA, $A^0(l)$ and $A(l)$, $l = 1, \dots, h$, associated to them would be

$$\boldsymbol{Q}^0(l) = \left\{Q_1^0(l), \dots, Q_{m^0(l)}^0(l)\right\}, \quad l = 1, \dots, h,$$

with $w^0(l) = m^0(l)$, and

$$\boldsymbol{Q}(l) = \left\{Q_1(l), \dots, Q_{m(l)}(l)\right\}, \quad l = 1, \dots, h.$$

Moreover we have the transition matrices $\boldsymbol{B}^0(l)$ and $\boldsymbol{B}(l) = [B_1(l) \mid B_2(l)]$, $l = 1, \dots, h$.

Now, for the models obtained joining the initial ones, we have the CJA given, see [6], by the Cartesian products $\mathrm{X}_{l=1}^h A^0(l)$ and $\mathrm{X}_{l=1}^h A(l)$.

With $\boldsymbol{Q}^*(l) = \{Q_1^*(l), \dots, Q_{m^*(l)}^*(l)\}$, $l = 1, \dots, h$, the principal basis of $A^*(l)$, the principal basis of $\mathrm{X}_{l=1}^h A^*(l)$ will be all the blockwise diagonal matrices with all the principal blocks null but one which will belong to one of the $\boldsymbol{Q}^*(l)$, $l = 1, \dots, h$.

Taking $\overline{m}^0(0) = 0$ and $\overline{m}^0(l) = \sum_{j=1}^{l} m^0(j)$, the matrices in the principal basis of $\mathrm{X}_{l=1}^h A^0(l)$ will be

$$\boldsymbol{Q}_i^0 = D\left(\mathbf{0}_{n_1 \times n_1} \dots \boldsymbol{Q}_{i-\overline{m}^0(l-1)}^0(l) \dots \mathbf{0}_{n_h \times n_h}\right), \quad \overline{m}^0(l-1) < i \leq \overline{m}^0(l),\ l = 1, \dots, h,$$

so we have the transition matrix

$$\boldsymbol{B}^0 = D\left(\boldsymbol{B}^0(1), \dots, \boldsymbol{B}^0(h)\right).$$

Going over to $\mathrm{X}_{l=1}^h A(l)$, to have the transition matrix

$$\boldsymbol{B} = [B_1 \mid B_2]$$

with

$$B_i = D\left(B_i(1), \dots, B_i(h)\right), \quad i = 1, 2$$

we take $v(l) = m(l) - z(l)$, $l = 1, \dots, h$, $\bar{v}(0) = 0$ and $\bar{v}(l) = \sum_{j=1}^{l} v(l)$, $l = 1, \dots, h$, as well as $\bar{z}(0) = 0$ and $\bar{z}(l) = \sum_{j=1}^{l} z(l)$, $l = 1, \dots, h$, to write

$$\boldsymbol{Q}_i = D\left(\mathbf{0}_{n_1 \times n_1} \dots \boldsymbol{Q}_{i-\bar{z}(l-1)}(l) \dots \mathbf{0}_{n_h \times n_h}\right), \quad \bar{z}(l-1) < i \leq \bar{z}(l),\ l = 1, \dots, h$$

and, with $\bar{z} = \bar{z}(h)$,

$$\boldsymbol{Q}_i = D\left(\mathbf{0}_{n_1 \times n_1} \dots \boldsymbol{Q}_{i-\bar{z}-\bar{v}(l-1)}(l) \dots \mathbf{0}_{n_h \times n_h}\right),$$
$$\bar{z} + \bar{v}(l-1) < i \leq \bar{z} + \bar{v}(l),\ l = 1, \dots, h.$$

We now establish

Proposition 1. *Joining models with commutative orthogonal block structure gives models with commutative orthogonal block structure.*

Proof. Since $\boldsymbol{X}_0^\dagger = D(\boldsymbol{X}_0^\dagger(1) \dots \boldsymbol{X}_0^\dagger(h))$, the orthogonal projection matrix $\boldsymbol{T}$ on $R(\boldsymbol{X}_0)$ will be

$$\boldsymbol{T} = \boldsymbol{X}_0\boldsymbol{X}_0^\dagger = D\big(\boldsymbol{X}_0(1)\boldsymbol{X}_0^\dagger(1) \dots \boldsymbol{X}_0(h)\boldsymbol{X}_0^\dagger(h)\big) = D\big(\boldsymbol{T}(1) \dots \boldsymbol{T}(h)\big),$$

which commutes with the $\boldsymbol{Q}_i^0$, $i = 1, \dots, m^0$, with $m^0 = \sum_{l=1}^h m^0(l)$. So the thesis is established. □

We also have the

Proposition 2. *Joining models with commutative orthogonal block structure with matching [segregation] gives models with commutative orthogonal block structure with matching [segregation].*

Proof. With $z^0(l)$, $l = 1, \dots, h$, $[z^0]$ the number of matrices that belong to both principal basis of $A^0(l)$ and $A(l)$, $l = 1, \dots, h$, $[\mathrm{X}_{l=1}^h A^0(l)$ and $\mathrm{X}_{l=1}^h A(l)]$ we have $z^0 = \sum_{l=1}^h z^0(l)$, so $z^0 = 0$, and there is matching for the joint model if and only if $z^0(1) = \dots = z^0(h) = 0$ and there is matching for all the initial ones.

Moreover the row vectors of $B_2 = D(B_2(1), \dots, B_2(h))$ are linearly independent, and there is segregation for the joint model if and only if the $B_2(l)$, $l = 1, \dots, h$, have linearly independent row vectors and there is segregation for all initial models. The proof is now complete. □

5. Final remarks

In order to analyse together models obtained independently, we can build up complex models from simple ones. For this purpose, two operations, models crossing and models nesting, were introduced, see [15]. These operations were based on the Kronecker product of CJA and the restricted Kronecker product of CJA, see [8].

In this paper we consider another operation, models joining, which, as we saw, is associated to the Cartesian product of Jordan Algebras, introduced in [6]. The relation between building of complex models from simple ones and operations on CJA is thus, once again, stressed.

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References

[1] R.A. Bailey, S.S. Ferreira, D. Ferreira, C. Nunes, Estimability of variance components when all model matrices commute, Linear Algebra Appl. 492 (2015) 144–160.
[2] T. Calinski, S. Kageyama, Block Designs: A Randomization Approach, vol. I: Analysis, Lecture Note in Statistics, vol. 150, Springer-Verlag, New York, 2000.
[3] T. Calinski, S. Kageyama, Block Designs: A Randomization Approach, vol. II: Design, Lecture Note in Statistics, vol. 170, Springer-Verlag, New York, 2003.
[4] F. Carvalho, J.T. Mexia, M.M. Oliveira, Canonic inference and commutative orthogonal block structure, Discuss. Math. Probab. Stat. 28 (2) (2008) 171–181.
[5] F. Carvalho, J.T. Mexia, C. Santos, C. Nunes, Inference for types and structured families of commutative orthogonal block structures, Metrika 78 (2015) 337–372.
[6] C. Fernandes, P. Ramos, J.T. Mexia, Algebraic structure of step nesting designs, Discuss. Math. Probab. Stat. 30 (2) (2010) 221–235.
[7] S. Ferreira, D. Ferreira, C. Nunes, J.T. Mexia, Estimation of variance components in linear mixed models with commutative orthogonal block structure, Rev. Colombiana Estadíst. 36 (2) (2013) 259–269.
[8] M. Fonseca, J.T. Mexia, R. Zmyslony, Binary operations on Jordan algebras and orthogonal normal models, Linear Algebra Appl. 117 (1) (2006) 75–86.
[9] M. Fonseca, J.T. Mexia, R. Zmyslony, Inference in normal models with commutative orthogonal block structure, Acta Comment. Univ. Tartu. Math. 12 (2008) 3–16.
[10] A.M. Houtman, T.P. Speed, Balance in designed experiments with orthogonal block structure, Ann. Statist. 11 (4) (1983) 1069–1985.
[11] P. Jordan, J. Von Neumann, E. Wigner, On an algebraic generalization of the quantum mechanical formulation, Ann. of Math. 35 (1) (1934).
[12] A.I. Khuri, T. Mathew, B.K. Sinha, Statistical Tests for Mixed Linear Models, Wiley, New York, 1998.
[13] S. Mejza, On some aspects of general balance in designed experiments, Statistica LII (2) (1992) 263–278.
[14] J.T. Mexia, Introdução à Inferência Estatística Linear, Centro de Estudos de Matemática Aplicada, Edições Lusofonas, 1995.
[15] J.T. Mexia, R. Vaquinhas, M. Fonseca, R. Zmyslony, COBS: segregation, matching, crossing and nesting, in: Latest Trends and Applied Mathematics, Simulation, Modelling, 4th International Conference on Applied Mathematics, Simulation, Modelling, ASM'10, 2010, pp. 249–255.
[16] J.A. Nelder, The analysis of randomized experiments with orthogonal block structure I. Block structure and the null analysis of variance, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 283 (1965) 147–162.
[17] J.A. Nelder, The analysis of randomized experiments with orthogonal block structure II. Treatment structure and the general analysis of variance, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 283 (1965) 163–178.
[18] C. Nunes, C. Santos, J.T. Mexia, Relevant statistics for models with commutative orthogonal block structure and unbiased estimator for variance components, J. Interdiscip. Math. 11 (4) (2008) 553–564.
[19] C. Santos, C. Nunes, J.T. Mexia, OBS, COBS and mixed models associated to commutative Jordan algebra, in: Proceedings of 56th Session of the International Statistical Institute, Lisbon, 2007.
[20] J.R. Schott, Matrix Analysis for Statistics, Jonh Wiley & Sons, New York, 1997.
[21] J. Seely, Linear spaces and unbiased estimation, Ann. Math. Stat. 41 (1970) 1725–1734.
[22] J. Seely, Minimal sufficient statistics and completeness for multivariate normal families, Sankhya 39 (1977) 170–185.
[23] R. Zmyślony, A characterization of best linear unbiased estimators in the general linear model, in: Mathematical Statistics and Probability Theory, vol. 2, 1978, pp. 365–373.